

Twist free energy

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Abstract. One may impose to a system with spontaneous broken symmetry, boundary conditions which correspond to different pure states at two ends of a sample. For a discrete Ising-like broken symmetry, boundary conditions with opposite spins in two parallel limiting planes, generate an interface and a cost in free energy per unit area of the interface. For continuum symmetries the order parameter interpolates smoothly between the end planes carrying two different directions of the order parameter. The cost in free energy is then proportional to L^{d-2} for a system of characteristic size L . The power of L is related to the lower critical dimension, and the coefficient of this additional free energy vanishes at the critical temperature. In this note it is shown within a loop expansion that one does find the expected behavior of this twist free energy. This is a preamble to the study of situations where the broken continuum symmetry is believed to be more complex, as in Parisi ansatz for the Edwards-Anderson spin glass.

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1 Introduction

Spontaneously broken symmetries are characterized by the existence of several possible pure states. If one imposes “twisted” boundary conditions, *i.e.* different pure states at two ends of the system, the free energy per unit volume will be slightly greater than the free energy corresponding to one single pure state over the whole system.

For a simple discrete symmetry, such as the Z_2 -symmetry of Ising-like systems, one may consider an (hyper)-cubic system with up spins in the $z = 0$ plane, down spins in the $z = L$ plane and for instance periodic boundary conditions in the transverse directions x_1, x_2, \dots, x_{d-1} . This will generate an interface in the system centered around some plane $z = z_0$ and a cost in free energy

$$\Delta F = F_{\uparrow,\downarrow} - F_{\uparrow,\uparrow} = \sigma L^{d-1} \quad (1)$$

in which the interfacial tension $\sigma(T)$ is finite at low temperature, but vanishes at the critical temperature as

$$\sigma(T) = \sigma_0 \left(\frac{T_c - T}{T_c} \right)^\lambda. \quad (2)$$

As is well-known the power $(d-1)$ of L in (1) implies that the lower critical dimension of systems with a discrete

symmetry is equal to one, *i.e.* there is no ordered phase unless d is greater than one. Widom [1] has first proposed the scaling law

$$\lambda = (d-1)\nu \quad (3)$$

in which ν is the correction length exponent

$$\xi = \xi_0 \left(\frac{T_c - T}{T_c} \right)^\nu. \quad (4)$$

The corresponding amplitude relation implies that the combination

$$\xi_0^{d-1} \sigma_0 \quad (5)$$

is universal. All this was studied long ago [2] by renormalization group techniques and $(4-d)$ -expansion. At leading order the classical (mean field) solution is a kink, of hyperbolic tangent shape, interpolating between up and down spins, and fluctuations are given at one-loop order by the Fredholm determinant of a one-dimensional Schrödinger operator in a $1/\cosh^2(z - z_0)$ potential which, as is well-known, is solvable analytically.

For more complex spontaneously broken symmetries, continuum symmetries, or replica-symmetry breaking, the situation is less trivial, and it is necessary to look into the problem in some detail in order to understand the lower critical dimension. For a continuum symmetry group G , broken down to a subgroup H , as in N -vector models, one considers the free energy with two different pure

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states in the two planes $z = 0$ and $z = L$. For an N -vector model one considers for instance an order parameter uniform along the vector $(1, 0, \dots, 0)$ in the $z = 0$ plane, and uniform but rotated by an angle θ_0 in the plane $z = L$, *i.e.* lying along the vector $(\cos \theta_0, \sin \theta_0, 0, \dots, 0)$. There again one expects a cost in free energy

$$\Delta F = \sigma(T, \theta_0)L^{d-2} \quad (6)$$

in agreement with a lower critical dimension equal to two, and with a “twist” energy $\sigma(T, \theta_0)$ (or spin stiffness constant) vanishing as θ_0^2 for small θ_0 , (the ratio σ/θ_0^2 is the helicity modulus defined by Fisher *et al.* [3]), and vanishing at T_c [4,5] like $(T_c - T)^{\nu(d-2)}$. If it is quite elementary to verify these statements within mean field theory, not difficult also to check them in the vicinity of the lower critical dimension $d_l = 2$ through the non-linear sigma model [6,7], it is not so simple to examine the problem below the upper critical dimension $d_u = 4$. This note is thus devoted to this point. Our aim in performing this calculation is to repeat it later for a spin glass. There, below the temperature of transition, one recovers a broken continuum symmetry, displayed by Parisi’s ansatz [8] of replica symmetry breaking for the Edwards-Anderson model, which yields a continuum of schemes with equal free energy (reparametrization invariance). If the situation at $d_u = 6$ is more or less under control, despite its complexity [9], the knowledge about d_l is poor, although it is believed to lie between two and three [9,10,12]. If one imposes again two different schemes at two ends of the system, one expects [11] a cost in free energy

$$\Delta F = \sigma L^{d-2+\eta} \quad (7)$$

with some negative anomaly η which would yield a lower critical dimension $d_l = 2 - \eta$. The possible presence of this anomaly requires to compute at least a one-loop correction to mean field theory. This difficult calculation will be reported in a subsequent article, and this note simply aims at showing that already for the well understood N -vector model, the theory is somewhat involved. The rest of this note is thus devoted to the N -vector model below four dimensions, treated through a $(\phi^2)^2$ field theory and an $\epsilon = 4 - d$ expansion. It is interesting to note that a direct calculation of the helicity modulus has also been performed directly in three dimensions, in spite of the singularities expected from Goldstone massless modes [15,16]. This has been done by keeping a symmetry breaking field until one can let it go safely to zero at the end of the calculation of this helicity modulus.

2 Mean field theory

The action for the N -vector model in the broken symmetry domain is

$$S = \int_0^L dz \int d^{d-1}x_\perp \left[\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}|t|(\phi)^2 + \frac{g}{4}((\phi)^2)^2 \right], \quad (8)$$

in which t is proportional to $T - T_c$. A pure state throughout the bulk would have a magnetization \mathbf{M} whose magnitude is given by

$$|t| = gM^2. \quad (9)$$

Subtracting the bulk contribution one thus has

$$\Delta S = \int_0^L dz \int d^{d-1}x_\perp \left[\frac{1}{2}(\nabla\phi)^2 + \frac{g}{4}(\phi^2 - M^2)^2 \right]. \quad (10)$$

The free energy ΔF is the value of the minimum of ΔS with the boundary conditions

$$\begin{aligned} \phi(z = 0, \mathbf{x}_\perp) &= M(1, 0, \dots, 0) \\ \phi(z = L, \mathbf{x}_\perp) &= M(\cos \theta_0, \sin \theta_0, 0, \dots, 0). \end{aligned} \quad (11)$$

We fix here the value of the order parameter on the edges, rather than imposing magnetic fields on the boundaries. Our partition function will thus be defined with fixed prescribed values of the order parameter on the two edges, rather than fixing a surface magnetic field and letting the surface order parameter fluctuate, as in the work of Krech [13] for instance. For an $N = 1$ (scalar) order parameter, we would have to fix the surface order parameter to a value slightly smaller than the bulk magnetization, but for $N > 1$ one can directly take the modulus of the surface order parameter equal to the bulk magnetization, as shown in the mean field solution of the equations of motion below. It is easy to verify that ΔS is minimum

- when the order parameter remains in the 2-plane of the two vectors defined by the boundary conditions
- when ϕ is a function of z -alone, *i.e.* independent of x_\perp .

and one can parametrize the mean field solution as

$$\phi = \rho(z)(\cos \theta(z), \sin \theta(z), 0, \dots, 0), \quad (12)$$

for which

$$\begin{aligned} \Delta S &= L^{d-1} \int_0^L dz \left[\frac{1}{2} \left(\frac{d\rho}{dz} \right)^2 \right. \\ &\quad \left. + \frac{1}{2}\rho^2 \left(\frac{d\theta}{dz} \right)^2 + \frac{g}{4}(\rho^2 - M^2)^2 \right]. \end{aligned} \quad (13)$$

The solution will be close to that of an order parameter uniformly rotating between the two planes with a constant magnitude M , namely $\rho(z) = M$ and $\theta(z) = \frac{z}{L}\theta_0$, for which $\Delta F = \frac{1}{2}\theta_0^2 M^2 L^{d-2}$. However, although the solution is close to that for large L , we shall need the corrections of order $1/L^2$ to that simple ansatz, and one has to solve the variational equations

$$\begin{aligned} \frac{d}{dz}(\rho^2\theta') &= 0 \\ \rho'' - \rho\theta'^2 - g\rho(\rho^2 - M^2) &= 0. \end{aligned} \quad (14)$$

Defining the dimensionless variables

$$\begin{aligned} \tau &= z\sqrt{2gM^2} \\ r(\tau) &= \frac{1}{M}\rho(z), \end{aligned} \tag{15}$$

the equations of motion are easily cast into the form

$$\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= \omega - v(r) \\ r^2 \frac{d\theta}{d\tau} &= \sqrt{\gamma}, \end{aligned} \tag{16}$$

with

$$v(r) = \frac{\gamma}{r^2} - \frac{1}{4}(1 - r^2)^2. \tag{17}$$

We can think of the equation for r as an equation of motion in (r, τ) -plane in which r starts at $r = 1$ for $\tau = 0$, decreases down to some r_0 , then increases and returns to $r = 1$ at

$$\tau_0 = L\sqrt{2gM^2}. \tag{18}$$

The parameters γ and ω have still to be determined by the boundary conditions. Of course the exact solution of the equations of motion (16) involves elliptic functions [13]. However it turns out that it is sufficient for our purpose to consider the regime in which γ is small, which corresponds to L large compared to the correlation length ξ (or if L/ξ is finite, corresponds to small θ_0). Indeed in that regime the order parameter has essentially a fixed length $r(\tau)$ close to 1, and

$$\begin{aligned} \frac{\theta_0^2}{2gM^2L^2} &= \gamma + O(\gamma^2) \\ \omega &= \gamma + O(\gamma^2) \\ r &= 1 - \gamma s(\tau) + O(\gamma^2). \end{aligned} \tag{19}$$

The full integration to this order in $\frac{\theta_0^2}{2gM^2L^2}$ is then easy and leads to

$$s(\tau) = 1 - \frac{\cosh\left|\frac{\tau_0}{2} - \tau\right|}{\cosh\left(\frac{\tau_0}{2}\right)}. \tag{20}$$

To that same order one finds

$$\Delta F = \frac{1}{2}\theta_0^2M^2L^{d-2} + O(\gamma^2) = \frac{1}{2g}\theta_0^2L^{d-2}|t| + O(\gamma^2). \tag{21}$$

Let us note that, in mean field, the correlation length is related to the temperature by

$$\xi^{-2} = 2gM^2 = 2|t|, \tag{22}$$

and thus γ is small either because L/ξ is large or because θ_0 is small. The result (21) is thus in agreement with our expectations

$$\Delta F = \sigma L^{d-2} \tag{23}$$

with $\sigma = \frac{1}{2g}\theta_0^2|t|$ vanishing at the critical temperature. We also verify the scaling law $\sigma(t) \sim |t|^{\nu(d-2)}$ (which is expected to be true for $d \leq 4$) in four dimensions at which $\nu = 1/2$ and $\nu(d-2) = 1$.

As far as mean field theory is concerned the picture is simple: for $\theta_0\xi/L$ small, the magnitude of the order parameter remains close to M over the whole sample, and its direction smoothly interpolates between the two end planes with a constant angle gradient. If we went beyond this simple picture in (20) it is because this will be needed in the loop expansion when we consider fluctuations around the mean field.

3 One loop corrections

We now go to dimension $d = 4 - \epsilon$ and work to first order in ϵ , which requires the calculation of one-loop fluctuations around mean field theory. Instead of an ultra-violet cut-off given by some lattice spacing, it turns out to be much more convenient, as often, to use dimensional regularization. The mean field solution is

$$\phi_c = Mr(z)(\cos\theta(z), \sin\theta(z), 0, \dots, 0), \tag{24}$$

with r and θ described in the previous section. It is convenient to introduce an orthonormal moving frame consisting of the vectors

$$\begin{aligned} \mathbf{e}_1 &= (\cos\theta(z), \sin\theta(z), 0, \dots, 0) \\ \mathbf{e}_2 &= (-\sin\theta(z), \cos\theta(z), 0, \dots, 0), \end{aligned} \tag{25}$$

plus the $(N - 2)$ fixed unit vectors $\mathbf{e}_a, (a = 3, \dots, N)$ perpendicular to the two-plane (1-2). The field $\phi(z, \mathbf{x}_\perp)$ is then parametrized as

$$\begin{aligned} \phi(z, \mathbf{x}_\perp) &= (\rho(z) + \psi_1(z, \mathbf{x}_\perp))\mathbf{e}_1(z) \\ &+ \psi_2(z, \mathbf{x}_\perp)\mathbf{e}_2(z) + \sum_{a=3}^N \psi_a(z, x_\perp)\mathbf{e}_a. \end{aligned} \tag{26}$$

The boundary conditions on those ψ_a are periodic in the transverse directions and, since the mean field order parameter ϕ_c is equal to the magnetization on the boundaries, one has to impose Dirichlet conditions on the fluctuating fields $\psi_a(z = 0) = \psi_a(z = L) = 0$. A one-loop calculation requires to keep only the quadratic terms in ψ_a of the action. Collecting those terms one finds

$$S = S_0 + S_2 \tag{27}$$

in which S_0 is the mean field contribution and

$$S_2 = \int_0^L dz \int d^{d-1}x_\perp \left[\frac{1}{2} \sum_1^N (\nabla\psi_a)^2 + \frac{1}{2} \left(2gM^2 + \left(\frac{d\theta}{dz} \right)^2 + 3gM^2(r^2(z) - 1) \right) \psi_1^2 + \frac{d\theta}{dz} \left(\psi_1 \frac{\partial\psi_2}{\partial z} - \psi_2 \frac{\partial\psi_1}{\partial z} \right) + \frac{1}{2} \left(\left(\frac{d\theta}{dz} \right)^2 + gM^2(r^2(z) - 1) \right) \psi_2^2 + \frac{1}{2} gM^2(r^2(z) - 1) \sum_3^N \psi_a^2 \right]. \quad (28)$$

The one-loop free energy is thus equal to the properly normalized

$$\Delta F = S_0 + \frac{1}{2} \text{Tr} \ln \frac{\partial^2 S_2}{\partial\psi_a(x) \partial\psi_b(y)}. \quad (29)$$

The normalization will be chosen such that ΔF vanishes with θ_0 .

Contribution of the N fluctuating modes

- The transverse modes ψ_a , $a = 3, \dots, N$ are decoupled and give a contribution to ΔF equal to

$$\frac{1}{2} (N-2) \text{Tr} \ln [-\nabla^2 + gM^2(r^2(z) - 1)] = \frac{1}{2} (N-2) \times \int \frac{d^{d-1}q_\perp}{(2\pi)^{d-1}} \text{Tr} \ln \left[q_\perp^2 - \frac{d^2}{dz^2} + gM^2(r^2(z) - 1) \right]. \quad (30)$$

One should note that although $r^2(z) - 1$ is negative, the spectrum of $-\frac{d^2}{dz^2}$ is bounded below by π^2/L^2 since we have Dirichlet boundary conditions on the planes $z = 0$ and $z = L$. Taking the explicit solution (19, 20) one sees that the spectrum of $-\frac{d^2}{dz^2} + gM^2(r^2(z) - 1)$ is bounded below by $(\pi^2 - \theta_0^2)/L^2$ and is thus positive.

Therefore *a priori* one has to compute the Fredholm determinant of a one-dimensional Schrödinger operator in the complicated potential $r^2(z)$. However for large L , perturbation theory gives very simply the answer since $r^2(z) - 1$ is of order $1/L^2$. This is to be contrasted with a localized Ising interface, for which there is no small parameter for large L . The simplification here is due to the fact that the order parameter turns slowly from one end of the system to the other one and thus has only small fluctuations in the moving frame that we have introduced.

Then we may replace $\text{Tr} \ln [q_\perp^2 - \frac{d^2}{dz^2} + gM^2(r^2(z) - 1)]$ (subtracted to vanish at $\theta_0 = 0$)

by $gM^2 \text{Tr} [q_\perp^2 - \frac{d^2}{dz^2}]^{-1} (r^2(z) - 1)$. Expanding on the basis of the Dirichlet eigenstates of $-\frac{d^2}{dz^2}$, the states $\sqrt{\frac{2}{L}} \sin(n\pi z/L)$, we obtain the contribution of these modes, in the large L limit, under the form

$$- (N-2) \theta_0^2 L^{d-3} \int \frac{d^{d-1}q_\perp}{(2\pi)^{d-1}} \frac{1}{L} \times \int_0^L dz s(z) \sum_{n=1}^{\infty} \frac{\sin^2(n\pi z/L)}{q_\perp^2 + (n\pi/L)^2}.$$

in which $s(z)$ is the explicit mean field correction (20). In the large L limit one can replace $s(z)$ by one, the sum over n by an integral which, combined with the integral over q_\perp , gives the integral $L \int d^d p/p^2$ which vanishes in dimensional regularization. Those modes have thus a vanishing contribution to the terms proportional to L^{d-2} of ΔF .

- We now come to the coupled ψ_1 - ψ_2 modes, using again that $r^2(z) - 1$ and $(d\theta/dz)^2$ are of order $1/L^2$. This allows one again to use a perturbation expansion about a massless ψ_2 -mode and a massive ψ_1 . After a lengthy, but elementary calculation, we obtain the contribution of these two modes to ΔF under the form of a sum of five terms:

$$\Delta F^{\text{one-loop}} = \frac{\theta_0^2}{2L^2} \text{Tr} \left(\frac{1}{-\nabla^2 + 2gM^2} + \frac{1}{-\nabla^2} \right) - \frac{\theta_0^2}{2L^2} \text{Tr} \left(3 \frac{1}{-\nabla^2 + 2gM^2} s(z) + \frac{1}{-\nabla^2} s(z) \right) - 2 \frac{\theta_0^2}{2L^2} \text{Tr} \left(\frac{1}{(-\nabla^2 + 2gM^2)(-\nabla^2)} \left(-\frac{\partial^2}{\partial z^2} \right) \right). \quad (31)$$

We leave the detail of the calculations to an appendix and simply report the result. We have computed the $1/\epsilon$ pole of this expression, for arbitrary L/ξ and obtained.

$$\Delta F = \frac{\theta_0^2}{2g} L^{d-2} |t| + \frac{3}{8\pi^2 \epsilon} \theta_0^2 L^{d-2} |t|^{1-\epsilon/2} \quad (32)$$

(we have kept it under this form since g and $|t|^{\epsilon/2}$ have the same dimension). In this expression we have kept the finite ($|t| \ln |t|$) term and neglected the non-logarithmic terms.

4 Renormalization and scaling

We first note that the pole in $1/\epsilon$ in (32) is independent of L/ξ , as it should, since the renormalizations are independent of this ratio. Next we note that the limit of ϵ going to zero should be finite, provided we perform a coupling constant and mass renormalization (there is no wave function renormalization at this one-loop order). Taking the standard one loop result from literature [14] (with the

appropriate normalization of the coupling constant chosen in (8)) one has, at one-loop,

$$\frac{1}{g} = \mu^{-\epsilon} \left(\frac{1}{g_R} - \frac{N+8}{8\pi^2\epsilon} \right) \quad (33)$$

for the coupling constant renormalization (μ is an arbitrary inverse length scale) and

$$t = t_R \left(1 + g_R \frac{N+2}{8\pi^2\epsilon} \right) \quad (34)$$

for the mass (*i.e.* temperature) renormalization. This gives a renormalized expression for ΔF in terms of g_R and t_R which is finite, as expected, when ϵ goes to zero:

$$\Delta F = \frac{\theta_0^2}{2g_R} L^{d-2} \mu^{-\epsilon} |t_R| \left(1 - \frac{3g_R}{8\pi^2} \ln \frac{|t_R|}{\mu^2} \right). \quad (35)$$

The scaling of the coefficient of $\theta_0^2 L^{d-2}$ in the critical region requires a replacement of g_R by the infra-red stable fixed point

$$g_R^* = \frac{8\pi^2\epsilon}{N+8} + O(\epsilon^2), \quad (36)$$

and the exponentiation

$$\left(1 - \frac{3g_R}{8\pi^2} \ln \frac{|t_R|}{\mu^2} \right) \rightarrow \left(\frac{|t_R|}{\mu^2} \right)^{3g_R^*/8\pi^2}. \quad (37)$$

Given that the correlation length exponent ν has the expansion

$$\nu = \frac{1}{2} + \frac{N+2}{4(N+8)}\epsilon + O(\epsilon^2), \quad (38)$$

one verifies to this order that

$$1 - \frac{3g_R^*}{8\pi^2} = \nu(d-2) \quad (39)$$

which does yield the expected scaling law for the vanishing of the twist energy at T_c in the $O(N)$ -model.

5 Final remarks

Although *a priori* more cumbersome than the calculation of the interfacial energy for a discrete symmetry, it turns out that, for a continuum symmetry, it is possible to compute the complicated Fredholm determinant of fluctuations around mean field theory by an expansion in powers of $1/L$ which was not available for an interfacial wall. The calculation involves a description of the mean field solution in which it is not sufficient to simply assume that the order parameter rotates with a constant angle gradient from end to end, with a fixed length equal to the magnetization. The calculation presented here may be easily generalized to any continuum symmetry group G , broken

down spontaneously below a critical temperature to a subgroup H , with an order parameter in a given irreducible representation R of G .

This calculation provides an explicit test of the fact that the renormalizations are the same around any background solution: in the usual case one expands about a classical solution which is constant over the sample, whereas here one expanded around a non-trivial solution, and yet we found that the same coupling constant and mass renormalizations did work. We have also verified that the finiteness of the end result of the free energy for any ratio L/ξ . However the method that we have followed, has made use of a small parameter, namely $\theta_0\xi/L$. Away from the critical temperature this parameter is small because L is large. However if L/ξ is finite our calculation is restricted to small θ_0 . In the finite L/ξ regime, ΔF is *a priori* a more complicated function of θ_0 for which we have only determined the first term. Let us stress also that we have used the ϵ -expansion, since we wanted to determine the behavior of the twist free energy near the upper critical dimension.

Appendix: One-loop divergences

Let us return to the five terms contained in (31) for the one-loop calculation of ΔF .

•

$$(a) = \frac{\theta_0^2}{2L^2} \text{Tr} \left(\frac{1}{-\nabla^2 + 2gM^2} \right). \quad (A.1)$$

If L goes to infinity first we may simply neglect the quantization of the longitudinal modes and write

$$\begin{aligned} (a) &= \frac{\theta_0^2}{2} L^{d-2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + 2gM^2} \\ &= -\frac{1}{16\pi^2\epsilon} \theta_0^2 L^{d-2} (2gM^2)^{1-\epsilon/2} \end{aligned} \quad (A.2)$$

in which it is understood that we have neglected the terms of order ϵ^0 . For finite L/ξ the calculation is much more involved. Going to the large L limit for the transverse periodic directions, but keeping the quantization of the longitudinal modes one has

$$\begin{aligned} (a) &= \frac{\theta_0^2}{2} L^{d-3} \int \frac{d^{d-1} q_\perp}{(2\pi)^{d-1}} \sum_{n=1}^{\infty} \frac{1}{q^2 + \frac{n^2\pi^2}{L^2} + 2gM^2} \\ &= \frac{\theta_0^2}{8\pi} L^{d-2} \int_0^\infty dq \frac{q^{d-2}}{\sqrt{q^2 + 2gM^2}} \\ &\quad \times \left[\coth L\sqrt{q^2 + 2gM^2} - \frac{1}{L\sqrt{q^2 + 2gM^2}} \right] \\ &= \frac{\theta_0^2}{8\pi} L^{d-2} (2gM^2)^{1-\epsilon/2} \int_0^\infty dx \frac{x^{d-2}}{\sqrt{x^2 + 1}} \\ &\quad \times \left[\coth l\sqrt{x^2 + 1} - \frac{1}{l\sqrt{x^2 + 1}} \right], \end{aligned} \quad (A.3)$$

in which

$$l = L\sqrt{2gM^2} = L/\xi. \quad (\text{A.4})$$

We now use the identity

$$\begin{aligned} \int_0^\infty dx \frac{x^{d-2}}{\sqrt{x^2+1}} \left[\coth l\sqrt{x^2+1} - \frac{1}{l\sqrt{x^2+1}} \right] = \\ \int_0^\infty dx \frac{x^{d-2}}{\sqrt{x^2+1}} \left[1 - \frac{1}{lx} \right] + \int_0^\infty dx \frac{x^{d-2}}{\sqrt{x^2+1}} \\ \times \left[\coth l\sqrt{x^2+1} - \frac{1}{l\sqrt{x^2+1}} - 1 + \frac{1}{lx} \right]. \quad (\text{A.5}) \end{aligned}$$

The first term of the r.h.s. of (A.5) is elementary and gives $-1/(2\epsilon)$ plus finite terms. It is easy to see that the second integral of the r.h.s. of (A.5) is finite when $d \rightarrow 4$. This proves that the divergent part of (a) is as expected independent of $l = L/\xi$.

$$(b) = \frac{\theta_0^2}{2L^2} \text{Tr} \left(\frac{1}{-\nabla^2} \right). \quad (\text{A.6})$$

Again if L goes to infinity first

$$(b) \rightarrow \frac{\theta_0^2}{2(2\pi)^d} L^{d-2} \int \frac{d^d p}{p^2} \quad (\text{A.7})$$

which vanishes in the dimensional regularization scheme.

$$\begin{aligned} (c) &= -\frac{3\theta_0^2}{2L^2} \text{Tr} \left(\frac{1}{-\nabla^2 + 2gM^2} s(z) \right) \\ &= -\frac{3\theta_0^2}{2L^2} \frac{L^{d-1}}{(2\pi)^{d-1}} \int d^{d-1} q_\perp \frac{2}{L} \\ &\quad \times \int_0^L dz \sum_1^\infty \left(\frac{\sin^2 n\pi z/L}{q_\perp^2 + n^2\pi^2/L^2 + 2gM^2} s(z) \right). \quad (\text{A.8}) \end{aligned}$$

Again if one lets L go to infinity first one can replace $s(z)$ by one, the calculations are then elementary and yield

$$(c) = \frac{3}{16\pi^2\epsilon} (2gM^2)^{1-\epsilon/2} \theta_0^2 L^{d-2}. \quad (\text{A.9})$$

For finite $L\xi$ one can prove with the help of the explicit form for $s(z)$ that the divergent part is unchanged.

$$(d) = -\frac{\theta_0^2}{2L^2} \text{Tr} \left(\frac{1}{-\nabla^2} s(z) \right). \quad (\text{A.10})$$

Again it is easy with the same integral representation

$$\begin{aligned} (d) &= -\frac{\theta_0^2}{2L^2} \frac{L^{d-1}}{(2\pi)^{d-1}} \int d^{d-1} q_\perp \frac{2}{L} \\ &\quad \times \int_0^L dz \sum_1^\infty \left(\frac{\sin^2 n\pi z/L}{q_\perp^2 + n^2\pi^2/L^2} s(z) \right), \quad (\text{A.11}) \end{aligned}$$

to prove that the leading term, proportional to L^{d-2} , multiplies the integral $\int d^d p/p^2$ which vanishes.

Therefore

$$(d) = 0 \quad (\text{A.12})$$

$$\begin{aligned} (e) &= -2 \frac{\theta_0^2}{2L^2} \text{Tr} \left(\frac{1}{(-\nabla^2 + 2gM^2)(-\nabla^2)} \left(-\frac{\partial^2}{\partial z^2} \right) \right) \\ &= -\frac{2\theta_0^2}{L^2} \frac{L^{d-1}}{(2\pi)^{d-1}} \int d^{d-1} q_\perp \\ &\quad \times \sum_1^\infty \left(\frac{(n\pi/L)^2}{(q_\perp^2 + n^2\pi^2/L^2)(q_\perp^2 + n^2\pi^2/L^2 + 2gM^2)} \right), \quad (\text{A.13}) \end{aligned}$$

which, in the large L limit, goes to

$$\begin{aligned} (e) &= -\frac{2\theta_0^2}{L^2} \frac{L^d}{(2\pi)^d} \int d^d p \frac{p_\perp^2}{p^2(p^2 + 2gM^2)} \\ &= -\frac{2\theta_0^2}{L^2} \frac{L^d}{d(2\pi)^d} \int d^d p \frac{1}{p^2 + 2gM^2}, \quad (\text{A.14}) \end{aligned}$$

from which one finds easily that

$$(e) = \frac{1}{16\pi^2\epsilon} (2gM^2)^{1-\epsilon/2} \theta_0^2 L^{d-2}. \quad (\text{A.15})$$

Collecting the results (a) to (e) we end up with (32) of the third section.

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